ITP–SB–93–61
19 October 1993

Principal Value Resummation

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ABSTRACT

We present a new resummation formula for the Drell-Yan cross section. The formal resummation of threshold corrections in Drell-Yan hard-scattering functions produces an exponent with singularities from the infrared pole of the QCD running coupling. Our reformulation treats such 'infrared renormalons' by a principal value prescription, analogous to a modified Borel transform. The resulting expression includes all large threshold corrections to the hard scattering function as an asymptotic series in α_s, but is a finite function of Q^2. We find that the ambiguities of the resummed perturbation theory imply the presence of higher twist corrections to quark-antiquark hard-scattering functions that begin at Λ_{QCD}/Q. This suggests an important role for higher twist in the phenomonolgy of hadron-hadron inclusive cross sections. We also discuss the numerical evaluation of the exponent and its asymptotic perturbation series for representative values of Q^2.

1. Introduction

It has been known for some time that perturbative QCD corrections to the inclusive Drell-Yan and other hadron-hadron hard scattering cross sections are numerically important, even at moderately high energies[1] . Considerable effort has been devoted to developing all-order resummation techniques in order both to study the convergence of the QCD perturbation series and to control the size of perturbative corrections to these processes. In a recent paper[2] , we presented an analysis of the resummed threshold corrections in the Drell-Yan cross section, deriving an explicit expression for the hard scattering function directly in momentum space. This new resummation formula organizes all large and order unity threshold corrections in the entire region where perturbation theory gives the dominant contribution. Because the resummed expression involves integrals over the scales of running couplings, it is undefined when these scales reach Λ_{QCD}. The purpose of the present work is to address these and related issues.

Beyond specific applications to dilepton cross sections, we are concerned here with some very general problems in the application of perturbative QCD to hard processes. A complete picture of such processes requires the inclusion of power-suppressed, or 'higher twist' contributions, but to include such contributions, we must first define the 'leading-twist', or perturbative series[3] [4] . To do so fully is a formidable task, beyond our present abilities. The class of large perturbative threshold corrections to dilepton (and other) inclusive hard scattering cross sections, however, is known to all orders. It is, as expected, ambiguous. Our goal, then, will be to 'make room' for higher twist by defining the resummation of large corrections at leading twist. At the same time, we may also ask why, despite all these large corrections and the ambiguities in perturbation theory, perturbative corrections at the one loop level are not totally wrong. While we cannot fully answer this question either, we will get some hints toward its resolution.

The situation for dilepton production hard-scattering functions is analogous, but not identical, to the situation for the e^+e^- annihilation total cross section[3] [4] . In e^+e^-, the ambiguities of perturbation theory may be identified with the gluon condensate $\langle 0|F^2|0\rangle$, which appears in the operator product expansion for this process. Such contributions

are suppressed by the power Q^{-4} relative to fixed-order perturbation theory. In contrast, we shall identify below the first ambiguity in the resummed perturbative cross section at the much larger level of Q^{-1} relative to leading power. This suggests an important phenomonological role for higher twist in hadron-hadron scattering, at least at moderate energies. Another difference between dilepton production and the e^+e^- annihilation cross section is that in the latter case low order corrections are much smaller. It is possible that the occurrence of nonperturbative corrections at lower twist in hadron-hadron scattering is related to the larger size of its perturbative corrections.

Let us now review the basic results of the resummation program applied to the dilepton cross section [1] [2]

$$h_1(p_1) + h_2(p_2) \to l\bar{l}(Q^\mu) + X \,, \tag{1.1}$$

with a produced lepton pair of momentum Q^μ. In perturbation theory, the factorized form of this cross section is given by

$$\frac{d\sigma}{dQ^2} = \sigma_0 \sum_{ab} \int_0^1 \frac{dx_a}{x_a}\frac{dx_b}{x_b} \phi_{a/h_1}(x_a, Q^2)\phi_{b/h_2}(x_b, Q^2)\omega_{ab}(\tau/x_a x_b, a_s(Q^2)) \,, \tag{1.2}$$

where σ_0 is the Born cross section, $\phi_{a/h_1}(x_a, Q^2)$ is the distribution function of parton a in hadron h_1, and ω_{ab} is a short-distance function, or 'hard part'. In eq. (1.2) we denote $\tau = Q^2/s$, with $s = (p_1 + p_2)^2$. We emphasize that when we speak of 'leading' and 'higher twist' in this paper, we are referring to the hard parts, and not to the cross section as a whole, which depends on the interplay of the short-distance functions with the evolving distributions.

In eq. (1.2) the hard parts ω_{ab}, calculable in perturbation theory, as well as the non-perturbative parton densities, are not unique. We shall identify the distribution ϕ_{a/h_i} with the contribution, F_{a/h_i}, of parton a to a structure function in deeply inelastic scattering of a nucleon h_i. Then the hard parts ω_{ab}, which directly give a prediction for the normalization of the Drell-Yan cross section in terms of the observables F_{a/h_i}, contain large corrections. These come from large numerical coefficients of δ-functions, and from 'plus distributions'. The organization of such large corrections to all orders in perturbation theory, is the content of various resummation techniques [5] [6] [7] [8] . It is convenient, in order to study the

large perturbative contributions in eq. (1.2) which come from the region $z=\tau/x_a x_b \to 1$ of the integrals, to take moments and study the large-n region. The only quantities that are singular in the $z\to 1$ limit are the diagonal-flavor quark-antiquark hard parts, for which we obtain the exponentiated form[2]

$$\tilde{\omega}_{q\bar{q}}(n,Q^2)=e_q^2 A(\alpha_s(Q^2))\tilde{I}(n,Q^2)\,, \tag{1.3}$$

where the terms that are singular in the limit $n\to\infty$ are contained in

$$\begin{aligned}\tilde{I}(n,Q^2) &= \exp\Big[-\int_0^1 dx\Big(\frac{x^{n-1}-1}{1-x}\Big)\Big\{\int_0^x \frac{dy}{1-y} g_1(\alpha_s[(1-x)(1-y)Q^2]) \\ &\qquad + g_2(\alpha_s[(1-x)Q^2])\Big\}\Big] \\ &\equiv \exp[E(n,Q^2)]\ .\end{aligned} \tag{1.4}$$

In eq. (1.3), $A(\alpha_s(Q^2))$ represents $\delta(1-z)$ contributions, including the exponentiated Sudakov π^2 terms[8] . $\tilde{I}(n,Q^2)$ contains the exponentiation of all plus-distributions, which are the source of growth with n. In ref. 2, we employed an asymptotic expansion for the exponent of eq. (1.4), to obtain, directly in momentum space, the resummed hard parts in the form

$$I(z,\alpha_s)=\delta(1-z)-\left[\frac{\mathrm{e}^{E\left(\frac{1}{1-z},\alpha_s\right)}}{\pi(1-z)}\Gamma\Big(1+P_1\Big(\frac{1}{1-z},\alpha_s\Big)\Big)\sin\Big(\pi P_1\Big(\frac{1}{1-z},\alpha_s\Big)\Big)\right]_+\,, \tag{1.5}$$

where

$$P_1(n,\alpha_s)\equiv\frac{\partial}{\partial\ln n}E(n,\alpha_s)\ . \tag{1.6}$$

The central issue that we did not address in ref. 2 was the nature of this asymptotic approximation and its relation to non-perturbative effects. This is the main subject of the present work.

Even though $\omega_{q\bar{q}}$ is finite at any finite order of perturbation theory, the exponentiated form, eqs. (1.4), (1.5), suffers from singularities associated with the behavior of the running coupling at small energy scales. Therefore, even though the above resummation formulas are useful in reproducing finite-order results[9] , a direct comparison with phenomenology

can be made only if a regularization of soft-gluon effects is supplied. Hence, a sensitivity of the resummed predictions to the treatment of this 'soft' region, $x \to 1$ in eq. (1.4), is to be expected[6] . For an infrared safe function, such regions give contributions that are finite order-by-order in perturbation theory, but which diverge in the sum to all orders. This behavior is, in fact, ubiquitous in QCD[3] [10] . In addition to its applications to the Drell-Yan and related cross sections, the resummation (1.5) may thus serve as a 'laboratory' to test our ideas about the asymptotic nature of the QCD perturbation series.

In this paper, we shall propose a finite definition of the exponent $E(n, Q^2)$ in eq. (1.4) as a *principal value* integral in the moment variable x. In this fashion we will define unambiguously - albeit not uniquely - the perturbative content of large corrections to the hard scattering function. We will see as well that any such definition requires that nonperturbative corrections to the hard scattering functions begin at order Q^{-1}, a full power of momentum transfer larger than in spin-averaged deeply inelastic scattering, and three powers larger than in the total e^+e^- annihilation cross section.

Let us reemphasize that our choice for a principal value resummation is not unique. The choice to be given below is closely linked to the specific form for the resummed hard part given in eq. (1.4). Other expressions for $\omega_{q\bar{q}}$, which sum the same sets of large perturbative contributions, but which differ in nonleading terms, are possible. For such expressions, a different use of the principal value, or even the use of some other prescription, may be advantageous. Differences in these prescriptions will be reflected in differences in higher twist corrections[3] [4] . In fact, there is a whole *class* of resummation formulas, which can be constructed by exploiting the analytic structure of the perturbative running coupling. In addition to studying our specific construction, we would like to argue that these issues bear further investigation in hadron-hadron scattering cross sections.

In section 2 we define our principal value resummation formula for $\omega_{q\bar{q}}(n, Q^2)$, and discuss its relation to the Borel transform. We relate the exponentiated principal value prescription to an analogous prescription for the evolution equation satisfied by $\omega_{q\bar{q}}(z, Q^2)$. We go on to evaluate analytically the leading (one-loop in the g_i and the running coupling) exponent in eq. (1.4). We then discuss the consequences of the principal value prescription for higher twist. Here we identify contributions at order $1/Q$, associated with the ambiguities of the resummed perturbation theory. In section 3, we construct an explicit asymptotic perturbation series for the exponent, and in section 4 we explore the exponent

and its asymptotic series numerically, again approximating it by the one-loop terms of the functions g_i. We find that a truncated perturbative series can give an excellent approximation to the exponent over a wide range in the variable n, but that such approximations require an energy-dependent maximum order, which differs for different terms in $E(n, Q^2)$. We shall also see that the full principal value exponent increases to a moderate value as n increases, and then turns over and decreases as $n \to \infty$. Finally, in section 5 we summarize our results and conclusions, and discuss the outlook for applying this method to the calculation of other hard hadronic processes.

2. The Principal Value Exponent

As explained in the introduction, we will construct a resummation formula analogous to eq. (1.4), that is calculable without reference to explicit IR cutoffs. We shall deal directly with the exponent $E(n, Q^2)$ given by (1.4). This defines the resummation in both moment and momentum space.

2.1 Definition

Denoting the expression in the curly brackets of eq. (1.4) by

$$\Gamma_{q\bar{q}}(1-x, Q^2) \equiv \int_0^x \frac{dy}{1-y} g_1(\alpha_s[(1-x)(1-y)Q^2]) + g_2(\alpha_s[(1-x)Q^2])\,, \tag{2.1}$$

we *define* the principal value exponent for the quark cross section in moment space through

$$\begin{aligned} E(n, Q^2) &= -\int_P d\zeta \Big(\frac{\zeta^{n-1}-1}{1-\zeta}\Big)\Gamma_{q\bar{q}}(1-\zeta, Q^2) \\ &\equiv -\frac{1}{2}\Big\{ \int_{0+i\epsilon}^{1+i\epsilon} d\zeta \Big(\frac{\zeta^{n-1}-1}{1-\zeta}\Big)\Gamma_{q\bar{q}}(1-\zeta, Q^2) + \int_{0-i\epsilon}^{1-i\epsilon} d\zeta \Big(\frac{\zeta^{n-1}-1}{1-\zeta}\Big)\Gamma_{q\bar{q}}(1-\zeta, Q^2)\Big\}\,. \end{aligned} \tag{2.2}$$

Wherever $\Gamma_{q\bar{q}}(1-\zeta, Q^2)$ is an analytic function between 0 and 1, we can deform the two contours back to the real axis. This is the case at any finite order in perturbation theory.

The resummed exponent, on the other hand, contains IR divergences near $x = 1$ *for any value of* n from the running of the coupling. In this region, the quark cross section defined through $E(n, Q^2)$ remains well defined with the principal value prescription, although, of course, contributions from this region are still infrared-sensitive.

How may we motivate the prinicpal value prescription? In some sense, the choice is quite arbitrary. But some choice is necessary if we are to 'make room' for higher-twist effects in physical situations where, as in dileption cross sections, perturbative corrections are large. In our opinion, it is precisely such quantities from which we may eventually learn the most about the interplay of perturbative and nonperturbative effects in QCD. With these observations in mind, we may offer a formulation of principal value resummation that is somewhat more general than the specific integrals in eq. (1.4).

The principal value prescription may be used to define a resummed series *whenever* the summation of a set of perturbative contributions can be expressed as an integral over the running coupling. Infrared renormalons[10] are the simplest examples of this form. This set of contributions is identified by a behavior from individual diagrams at nth order of $\alpha_s^n(Q^2)b_2^n n!$, with Q^2 any fixed scale. Such a series of terms may be generated by expanding the running coupling $\alpha_s(k^2)$ inside the simple integral

$$I(\alpha_s(Q^2)) \equiv \int_0^{Q^2} dk^2 k^2 \alpha_s(k^2) \tag{2.3}$$

as a power series in $\alpha_s(Q^2)$, using the one-loop expression,

$$\alpha_s(k^2) = \frac{\alpha_s(Q^2)}{1 + (b_2/\pi)\alpha_s(Q^2)\ln(k^2/Q^2)} = \frac{\pi}{b_2 \ln(k^2/\Lambda^2)} \,. \tag{2.4}$$

Many of the properties of infrared renormalons are brought out very clearly by reexpressing the series as a Borel transform,

$$\tilde{I}(b) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d(1/\alpha_s(Q^2)) e^{-ib/\alpha_s(Q^2)} I(\alpha_s(Q^2)) \,, \tag{2.5}$$

whose formal inverse is

$$I(\alpha_s(Q^2)) = \int_0^\infty db\, e^{-b/\alpha_s(Q^2)}\tilde{I}(b)\,. \tag{2.6}$$

The integral $I(\alpha_s(Q^2))$ is undefined as it stands, because of the singularity in the running coupling $\alpha_s(k^2)$ at $k^2 = \Lambda^2$. This singularity is directly reflected in the $n!$ behavior of the expansion in $\alpha_s(Q^2)$, and in a consequent singularity on the positive real axis in the Borel transform, $\tilde{I}(b)$, at $b = 2\pi/b_2$. In Ref. 3, it was shown that, given what we know now about the perturbative series, this is the leftmost singularity of the transform. Assuming the inverse transform, eq. (2.6), has been defined, such a singularity may be expected to contribute only at the level Q^{-4} to the cross section, corresponding exactly to the contribution of the gluon condensate, $\langle 0|\, F^{\mu\nu}F_{\mu\nu}\, |0\rangle$. One way to define the perturbative, and hence nonperturbative, contribution at this level is to define the inverse Borel transform as a principal value[3] . In fact, for the one-loop running coupling the resulting expression is *exactly* what we find by treating the original integral in eq. (2.3) as a principal value in k^2, and by changing variables to

$$b' \equiv \ln(k^2/Q^2)\,. \tag{2.7}$$

A similar change allows us to reproduce two-loop results as well.

In the somewhat more complex situation of eq. (1.4), integrals of the running coupling appear in an exponent, and the relation of a principal value in the moment integrals to an inverse Borel transform for $I(n,Q^2)$ is not quite so simple. The prescription can still be given a reasonably general motivation, however, in an evolution equation satisfied by $\omega_{q\bar{q}}$.

Up to corrections that are finite in the $z \to 1$ limit, the dilepton hard-scattering function $\omega_{q\bar{q}}(z,Q^2)$ obeys an evolution equation of the form[5]

$$\begin{aligned}&\Big[\frac{\partial}{\partial\ln(1/(1-z))} - \frac{1}{2}\beta(g)\frac{\partial}{\partial g} + 1\Big]\omega_{q\bar{q}}(z,Q^2)\\ &\qquad = -\int_z^1 dy\,\Big[\frac{g_1(\alpha_s((1-y)^2Q^2))}{(1-y)}\Big]_+ \omega_{q\bar{q}}(y-z,Q^2) + g_2(\alpha_s(Q^2))\omega_{q\bar{q}}(z,Q^2)\,,\end{aligned} \tag{2.8}$$

whose solution is

$$\omega_{q\bar{q}}(z,Q^2)=\sum_{n=0}^{\infty}\prod_{i=0}^{n}\int_0^1 dx_i\Big\{\Big[\frac{1}{1-x_i}\int_0^x dy_i\frac{g_1(\alpha_s((1-x_i)(1-y_i)Q^2))}{1-y_i}\Big]_+ + \Big[\frac{g_2(\alpha_s((1-x_i)Q^2))}{(1-x_i)}\Big]_+\Big\}\delta(1-z-\sum_{k=1}^{n}(1-x_k))\,. \tag{2.9}$$

Moments of this expression with respect to n yield, up to corrections of order $1/n$, $\omega_{q\bar{q}}(n,Q^2)$ in eqs. (1.3) and (1.4). To verify that eq. (2.9) satisfies (2.8), we may convert the derivative of the delta function with respect to $\ln(1-z)$ into a sum of derivatives with respect to the x_k, and then integrate by parts. The remainder of the reasoning only requires the definitions of the running coupling and the plus distribution.

In terms of the formulation for $\omega_{q\bar{q}}$ just given, we see that the singularity in the running coupling is already present in the evolution equation, (2.8). If we choose to define the y integral of this equation as a principal value, its solution (2.9) is then given as a sum of products of principal value x_i integrals, and its moments become exponentials of principal value integrals. (The delta functions may be considered as referring to the real parts of the x_i.) We may, therefore, wish to consider our principal value prescription for the exponentiated resummation as grounded in a principal value definition of the integro-differential equation whose solution it is.

Some basic properties of the principal value resummation formula are immediately obvious. First, the exponentiated cross section in moment space, eq. (1.4), is real (for real n). This is because the contour P in eq. (2.2) is a sum of two mirror-symmetric contours with respect to the real axis, and the integrand is real on some portion of the real axis. Also, E in eq. (1.4) is finite by the definition of $\Gamma_{q\bar{q}}$, eq. (2.1).

At this point, we may come back to the Borel transformation[3] . All large perturbative corrections[2] are contained in those pieces of the function $\Gamma_{q\bar{q}}(1-\zeta,Q^2)$, eq. (2.1), whose dependence on ζ and Q^2 is of the form

$$\Gamma_{q\bar{q}}(1-\zeta,Q^2)=\sum_{k=0}^{2}\alpha^k\Gamma^{(k)}_{q\bar{q}}(\alpha\ln[1/(1-\zeta)])\,, \tag{2.10}$$

where we define

$$\alpha \equiv \alpha_s(Q^2)/\pi\,, \tag{2.11}$$

and where each $\Gamma^{(k)}$ is an infinite series in its arugment. In our exponent

$$E(n,Q^2) = -\sum_{k=0}^{2} \alpha^k \int_P d\zeta \Big(\frac{\zeta^{n-1}-1}{1-\zeta}\Big)\Gamma^{(k)}_{q\bar{q}}(\alpha \ln[1/(1-\zeta)])\ , \tag{2.12}$$

we now expand the power of ζ,

$$\zeta^{n-1} = (1-(1-\zeta))^{n-1} = \sum_{m=0}^{\infty}(1-n)...(1-n+m-1)\frac{1}{m!}(1-\zeta)^m = \sum_{m=0}^{\infty}\frac{(1-n)_m}{m!}(1-\zeta)^m \tag{2.13}$$

where 'Pochhammer's symbol'[11] is defined as $(a)_m \equiv \Gamma(a+m)/\Gamma(a)$. Performing the change of variables

$$w = m\alpha \ln[1/(1-\zeta)] \tag{2.14}$$

we arrive at the following form for the exponent:

$$E(n,Q^2) = -\frac{1}{\alpha}\sum_{k=0}^{2}\alpha^k \sum_{m=1}^{\infty}\frac{(1-n)_m}{m!m}\gamma^{(k)}_{q\bar{q}}(\alpha;m)\ . \tag{2.15}$$

Here, $\gamma^{(k)}_{q\bar{q}}$ is given in terms of $\Gamma^{(k)}_{q\bar{q}}$, eq. (2.10), as

$$\gamma^{(k)}_{q\bar{q}}(\alpha;m) \equiv \int_{P'} dw \mathrm{e}^{-w/\alpha}\,\Gamma^{(k)}_{q\bar{q}}(w/m)\ , \tag{2.16}$$

where the principal value contour P' runs between 0 and ∞. This definition of $\gamma^{(k)}_{q\bar{q}}$ exactly coincides with the inverse Borel transform, defined as a principal value[3] .

2.2 Evaluation of the Exponent

We are now ready to discuss the explicit evaluation of the function $E(n,Q^2)$, eq. (2.2), which is defined in terms of the functions g_i in (2.1). The functions g_i are simple expansions

in terms of the running coupling constant $\alpha_s[\lambda Q^2]$, with λ a scale,

$$g_i(\alpha_s[\lambda Q^2]) = \sum_{j=1}^{\infty} \Big(\frac{\alpha_s[\lambda Q^2]}{\pi} \Big)^j g_i^{(j)} \ . \tag{2.17}$$

The lowest order numerical coefficients, that determine the large perturbative behavior, are[5] [6]

$$g_1^{(1)} = 2C_F, \quad g_2^{(1)} = -\frac{3}{2} C_F, \quad g_1^{(2)} = C_F \Big[C_A \Big(\frac{67}{18} - \frac{\pi^2}{6} \Big) - \frac{5n_f}{9} \Big] \ , \tag{2.18}$$

where n_f is the number of flavors. We can relate the resummed expression to one containing the fixed coupling $\alpha_s \equiv \alpha_s(Q^2)$, by reexpressing the running coupling at the scaling variables $(1-\zeta)(1-y)$ and $(1-\zeta)$ in terms of the fixed one, through the beta function. As remarked in ref. 2, *all* the large perturbative corrections in the resummation, of order unity or greater in the range $\alpha_s \ln[1/(1-z)] < 1$, are contained in terms that have at least as many powers of the logarithm of the momentum scale as of the fixed coupling. This means, in turn, that we only need to keep terms that are leading and next-to-leading in the g_i and the QCD beta-function. We now construct these terms explicitly.

Consider the renormalization-group equation for the running coupling $\alpha \equiv \alpha_s/\pi$,

$$\frac{\partial}{\partial \ln(\lambda)} \alpha[\lambda Q^2] = -b_2 (\alpha[\lambda Q^2])^2 - b_3 (\alpha[\lambda Q^2])^3 \tag{2.19}$$

where, in QCD,

$$b_2 = (33 - 2n_f)/12 \ , \quad b_3 = (306 - 38 n_f)/48 \ . \tag{2.20}$$

The solution of eq. (2.19), with the initial condition $\alpha(\lambda)|_{\lambda=1} = \alpha \equiv \alpha_s(Q^2)/\pi$, is

$$\alpha(\lambda)/\alpha = \Big[1 + \alpha b_2 \ln(\lambda) - (\alpha b_3/b_2) [\ln(\alpha(\lambda))/\alpha) - \ln((b_2/b_3 + \alpha(\lambda))/(b_2/b_3 + \alpha)) \Big]^{-1} \ . \tag{2.21}$$

We can solve this transcendental equation iteratively, keeping only leading ($\alpha^k \ln^k \lambda$) and

next-to-leading ($\alpha^k \ln^{k-1}\lambda$) powers. These terms are given by

$$\alpha(\lambda)/\alpha = \frac{1}{1+\alpha b_2 \ln\lambda} - (\alpha b_3/b_2)\frac{\ln(1+\alpha b_2\ln\lambda)}{(1+\alpha b_2\ln\lambda)^2}\ . \tag{2.22}$$

Combining eqs. (2.17) and (2.22) we find

$$g_i(\alpha[\lambda Q^2]) = \sum_{j=1}^{\infty} g_i^{(j)}\alpha^j\Big(\frac{1}{1+\alpha b_2\ln\lambda} - (\alpha b_3/b_2)\frac{\ln(1+\alpha b_2\ln\lambda)}{(1+\alpha b_2\ln\lambda)^2}\Big)^j\ . \tag{2.23}$$

The leading and next-to-leading terms, which give all large perturbative corrections, will come from the $j = 1,\ 2$ terms only. Keeping just the leading and next-to-leading pieces in these terms, we can write $g_i \simeq g_i^L + g_i^{NL}$ with

$$g_i^L(\alpha[\lambda Q^2]) = g_i^{(1)}\alpha\frac{1}{1+\alpha b_2\ln\lambda} \tag{2.24}$$

$$g_i^{NL}(\alpha[\lambda Q^2]) = -g_i^{(1)}\alpha^2(b_3/b_2)\frac{\ln(1+\alpha b_2\ln\lambda)}{(1+\alpha b_2\ln\lambda)^2} + g_i^{(2)}\alpha^2\frac{1}{(1+\alpha b_2\ln\lambda)^2}\ . \tag{2.25}$$

Note that the distinction between 'L' and 'NL' refers only to the loop order in g_i, and not to the powers of logarithms of n in $E(n,Q^2)$. In fact, $g_1^{(2)}$, for instance, gives the same maximum power of logarithms of n as $g_2^{(1)}$. For now, however, it will be instructive, and simpler, to focus on the one-loop terms only.

Let us now proceed with the analytic evaluation of the exponent

$$E(n,\alpha) = -\int_P \Big(\frac{\zeta^{n-1}-1}{1-\zeta}\Big)\Gamma_{q\bar{q}}(1-\zeta,Q^2) \simeq E(n,\alpha)_L + E(n,\alpha)_{NL}\ . \tag{2.26}$$

Here and below, we replace Q^2 by α as the second argument of E. E_L, as specified by (2.1) and (2.24), is

$$E(n,\alpha)_L = \alpha(g_1^{(1)}I_1 - g_2^{(1)}I_2)\,, \tag{2.27}$$

with

$$I_1(t) \equiv t\int_P d\zeta\Big(\frac{\zeta^{n-1}-1}{1-\zeta}\Big)\ln\Big(\frac{1+(2/t)\ln(1-\zeta)}{1+(1/t)\ln(1-\zeta)}\Big) = 2I(t/2) - I(t)\ , \tag{2.28}$$

$$I(t) \equiv t \int_P d\zeta \Big(\frac{\zeta^{n-1}-1}{1-\zeta}\Big) \ln(1+(1/t)\ln(1-\zeta))\,, \tag{2.29}$$

and

$$I_2(t) \equiv \int_P d\zeta \Big(\frac{\zeta^{n-1}-1}{1-\zeta}\Big) \frac{1}{1+(1/t)\ln(1-\zeta)}\ . \tag{2.30}$$

Here we define

$$t \equiv 1/(\alpha b_2) = \ln(Q^2/\Lambda^2)\,, \tag{2.31}$$

where we have used the one-loop running coupling, as in eq. (2.24).

The ζ integrals in I_1 that result from the expansion (2.13) are readily carried out by a change of variables to $\ln(1-\zeta)$, followed by integration by parts, which gives

$$I_1 = \sum_{m=1}^{\infty} \frac{(1-n)_m}{m!m^2} \{\mathcal{E}(mt) - 2\mathcal{E}(mt/2)\}\,, \tag{2.32}$$

where we define the combination

$$\mathcal{E}(x) \equiv x\mathrm{e}^{-x} Ei(x)\ ,\ \ x > 0\ , \tag{2.33}$$

with the Exponential Integral defined as the principal value integral[11]

$$Ei(x) \equiv \mathcal{P} \int_{-\infty}^{x} dy \frac{\mathrm{e}^y}{y}\ . \tag{2.34}$$

Similarly we find

$$I_2 = \sum_{m=1}^{\infty} \frac{(1-n)_m}{m!m} \mathcal{E}(mt)\ . \tag{2.35}$$

Hence the leading exponent can be written as

$$E(n,\alpha)_L = -\alpha g_1^{(1)} \sum_{m=1}^{\infty} \frac{(1-n)_m}{m!m^2} \Big[2\mathcal{E}(mt/2) - \mathcal{E}(mt)\Big] - \alpha g_2^{(1)} \sum_{m=1}^{\infty} \frac{(1-n)_m}{m!m} \mathcal{E}(mt)\ . \tag{2.36}$$

Notice in the above formulas that the moment dependence is entirely contained within the Pochhammer symbol, and is defined for complex n. The $\alpha_s(Q^2)$-dependence, on the other hand, is contained within the functions $\mathcal{E}(mt)$, which are defined for *any* value of t (α_s), however small (large).

2.3 Higher Twist in the Resummed Exponent

Eq. (2.36) affords a direct estimate of the ambiguities implicit in the pertrubative series. They are, as expected, of higher twist. We have treated these ambiguities by the prinicpal value prescription in eq. (2.34). Any other prescription for defining the integrals would differ in the treatment of the singularity at $y = 0$. But the integral at $y = 0$ is proportional to an exponential of $-t$, and is hence suppressed by a power of Λ/Q. The relevant powers may simply be read off from the arguments of the $\mathcal{E}$ in eq. (2.36). The minimum supression is from $\mathcal{E}(t/2)$ in the $m = 1$ term, which behaves as $(\Lambda/Q)\ln(Q^2/\Lambda^2)$. At this first nonleading power, perturbation theory is already ambiguous, and nonperturbative effects must come into play.

An alternative to the principal value prescription is to simply cut off the ζ integral in eq. (2.1) at a value large enough to avoid the singularities of the running coupling. It is perhaps worthwile to illustrate the relationship between these two approaches. In ref. 6. it was shown that the IR cutoff dependence in the hard part, despite being numerically significant, is higher-twist, although the precise powers were not determined. These powers may be easily determined if we take eq. (2.36) as a starting point, redefining the integrals that define the $\mathcal{E}$'s to reflect the cut-off.

For example, consider an exponent defined by cutting off the ζ integral at some minimum value of $1 - \zeta \equiv \xi$. For the integral I_2, for example, the corresponding regulated expression would be

$$I_2^{(reg)} = \sum_{m=1}^{\infty} \frac{(1-n)_m}{m!} I_2^{(reg)m} \tag{2.37}$$

with

$$I_2^{(reg)m} \equiv \int_{\xi_{min}}^{1} d\xi \xi^{m-1} \frac{1}{1 + (1/t)\ln\xi} = t\xi_0^m \int_{m\ln(\xi_{min}/\xi_0)}^{m\ln(1/\xi_0)} dx \frac{\mathrm{e}^x}{x} \ . \tag{2.38}$$

In the second form, $\xi_0 = \exp(-t)$ is the position of the pole in ξ. Since ξ_{min} is designed to make the integral finite, it must be chosen to be larger than the location of the pole. To separate only the nonperturbative region, however, we take it to be the same order of

magnitude as ξ_0. We parameterize the relation by

$$a \equiv \ln(\xi_{min}/\xi_0)\,, \tag{2.39}$$

with $0 < a < 1$. Then, using eqs. (2.35) and (2.38) we obtain:

$$I_2^m - I_2^{(reg)m} = t\mathrm{e}^{-mt}\mathcal{P}\int_{-\infty}^{ma} dx\frac{\mathrm{e}^x}{x} = t\mathrm{e}^{-mt}Ei(ma)\ . \tag{2.40}$$

For a given a, the difference between the cutoff integral and the principal value integral is exponentially suppressed by the *same* function of the fixed coupling constant that appears in the principal value expression (i.e., by the m-th power of Λ^2/Q^2). Exactly analogous results hold for I_1.

3. The Asymptotic Series

Even though eq. (2.36) gives the result for the exponent in a form appropriate for numerical evaluation, it is also of interest to study the asymptotic expansion of E in $\alpha_s(Q^2)$. Notice that the exponential integrals in eq. (2.36) have a perfectly well-defined Taylor expansion [11]

$$Ei(mt) = \gamma + \ln(mt) + \sum_{n=1}^{\infty}\frac{(mt)^n}{n!n}\,, \tag{3.1}$$

which actually converges better than an exponential for any value of mt. On the other hand, such a Taylor expansion would reproduce an infinite series of *inverse powers* of α_s. The issue we would like to address here is how to recover a perturbative series for the exponent, eq. (2.36).

We can obtain for the special function $\mathcal{E}(x)$, eq. (2.33), after repeatedly integrating by parts, the asymptotic expression

$$\mathcal{E}(mt) \simeq \sum_{\rho=0}^{N}\frac{\rho!}{(mt)^\rho}\,, \tag{3.2}$$

with N chosen to optimize the approximation. In this asymptotic series, we see explicitly the $\rho!$ behavior at ρth order, characteristic of infrared renormalons [10]. Their presence is a direct consequence of the singularity in the perturbative running coupling.

As we shall show in section 4, a common optimum N for all m can be determined numerically for each of the three sums in eq. (2.36). However, the three optimum numbers of asymptotic terms differ for the three sums, so we will use the (clumsy but we hope clear) notation $N \equiv \{N[2I(t/2)],\ N[I(t)],\ N[I_2(t)]\}$ for the three sums. Using the asymptotic expansion eq. (3.2) in the expression for E_L, eq. (2.36), the leading exponent becomes a finite perturbative sum:

$$
\begin{aligned}
E(n,\alpha,N)_L = -\alpha g_1^{(1)} \Bigg\{ & \sum_{\rho=0}^{N[2I(t/2)]} \rho! 2^{\rho+1} (\alpha b_2)^\rho \sum_{m=1}^{\infty} \frac{(1-n)_m}{m! m^{\rho+2}} \\
& - \sum_{\rho=0}^{N[I(t)]} \rho! (\alpha b_2)^\rho \sum_{m=1}^{\infty} \frac{(1-n)_m}{m! m^{\rho+2}} \Bigg\} - \alpha g_2^{(1)} \sum_{\rho=0}^{N[I_2(t)]} \rho! (\alpha b_2)^\rho \sum_{m=1}^{\infty} \frac{(1-n)_m}{m! m^{\rho+1}} \ .
\end{aligned}
\tag{3.3}
$$

We can reexpress the infinite series in the moment variable in terms of a plus-distribution, through the identity

$$
\sum_{m=1}^{\infty} \frac{(1-n)_m}{m! m^{\rho+1}} = \frac{(-1)^\rho}{\Gamma(\rho+1)} \int_0^1 dx x^{n-1} \left(\frac{\ln^\rho(1-x)}{1-x} \right)_+ \ . \tag{3.4}
$$

These integrals can in turn be expressed, for large n, as polynomials in $\ln n$. Using the relation

$$
\ln^\rho(1-x) = \lim_{\epsilon \to 0^+} \left(\frac{\partial}{\partial \epsilon} \right)^\rho (1-x)^\epsilon \,, \tag{3.5}
$$

we obtain

$$
\sum_{m=1}^{\infty} \frac{(1-n)_m}{m! m^{\rho+1}} = \lim_{\epsilon \to 0^+} \frac{(-1)^\rho}{\Gamma(\rho+1)} \left(\frac{\partial}{\partial \epsilon} \right)^\rho \left\{ \mathrm{B}(n,\epsilon) - \frac{1}{\epsilon} \right\} , \tag{3.6}
$$

where B is the Beta function. Using Stirling's formula for large n, we find

$$
\mathrm{B}(n,\epsilon) \simeq \mathrm{e}^{-\epsilon \ln n} \Gamma(\epsilon) \ , \tag{3.7}
$$

and we can approximate the infinite series in the moment variable, n, by a polynomial in

$\ln n$:

$$\sum_{m=1}^{\infty} \frac{(1-n)_m}{m! m^{\rho+1}} = (-1)^\rho \sum_{j=0}^{\rho+1} c_{\rho+1-j} \frac{(-1)^j}{j!} \ln^j n \ . \tag{3.8}$$

The numbers c_k above are the standard coefficients of the Taylor expansion[11] ,

$$\Gamma(1+z) = \sum_{k=0}^{\infty} c_k z^k \, . \tag{3.9}$$

To summarize, the leading exponent may be approximated by a finite perturbative sum $E(n,\alpha,N)_L$,

$$E(n,\alpha)_L = E(n,\alpha,N)_L + \Delta(n,\alpha,N)_L \, , \tag{3.10}$$

where the second term on the right is a remainder which, since it is not expressible in terms of powers of α_s, contains a higher-twist contribution. We shall discuss the numerical size of the remainder in section 4.

The truncated expansion $E(n,\alpha,N)_L$ defines a convergent, resummed perturbative series for the exponent E, and hence for the hard-scattering function. N is defined in eq. (3.10), to minimize Δ_L. Increasing N further will result in a smaller accuracy (larger Δ_L). Inversely, for a given accuracy, there is a *maximum* α_s beyond which the approximation of eq. (3.10) breaks down *for any* N, i.e., the remainder is outside the desired accuracy. These features of the exponent represent a 'nonperturbative barrier' in the accuracy of the asymptotic approximation, which is what we might expect.

The leading asymptotic exponent is given by eq. (3.8), in (3.3), as

$$\begin{aligned} E(n,\alpha,N)_L = & \sum_{\rho=1}^{N[2I(t/2)]+1} \alpha^\rho \sum_{j=0}^{\rho+1} s^L_{j,\rho}[2I(t/2)] \ln^j n \ + \sum_{\rho=1}^{N[I(t)]+1} \alpha^\rho \sum_{j=0}^{\rho+1} s^L_{j,\rho}[I(t)] \ln^j n \\ & + \sum_{\rho=1}^{N[I_2(t)+1]} \alpha^\rho \sum_{j=0}^{\rho} s^L_{j,\rho}[I_2(t)] \ln^j n \end{aligned} \tag{3.11}$$

with

$$s^L_{j,\rho}[2I(t/2)] = -g_1^{(1)} b_2^{\rho-1}(-1)^{\rho+j}\frac{(\rho-1)!}{j!}2^\rho c_{\rho+1-j} \ ,$$
$$s^L_{j\rho}[I(t)] = g_1^{(1)} b_2^{\rho-1}(-1)^{\rho+j}\frac{(\rho-1)!}{j!}c_{\rho+1-j} \ , \tag{3.12}$$
$$s^L_{j,\rho}[I_2(t)] = g_2^{(1)} b_2^{\rho-1}(-1)^{\rho+j}\frac{(\rho-1)!}{j!}c_{\rho-j} \ .$$

Again, the N's to be kept in the asymptotic expansion are fixed by minimizing the remainder in eq. (3.2) for a fixed value of $\alpha_s(Q^2)$. We may in principle use eqs. (3.11) and (3.12)(along with the corresponding results for E_{NL}) to approximate the resummed normalization by an analytic expression[2] . Alternately, we may use the full principal-value exponents. Using either the exact E, or its representation as an asymptotic series, the large perturbative corrections[2] to the hard scattering function in momentum space can be found from eqs. (1.5), (1.6).

4. Behavior of the Exponent

In this section we shall address several important numerical issues. Using the 'leading' exponent $E(n,\alpha)_L$, we will illustrate the behavior of the principal value definition and its approximation as an asymptotic series, $E(n,\alpha,N)_L$. We defer a similar discussion for the next-to-leading exponent, $E(n,\alpha,N')_{NL}$, as well as numerical results for cross sections, to future work. We will see that the principal value exponent behaves in a relatively mild fashion for all n, over a wide range of Q, and that the asymptotic series can give a good approximation to it unless n is very large.

In the following, we present numerical results for three values of Q, taking $\Lambda = 200 MeV$. To be specific, we have used the values

$$\alpha(Q=5GeV) = 0.075 \ , \quad \alpha(Q=10GeV) = 0.061 \ , \quad \alpha(Q=90GeV) = 0.039$$

along with[2]

$$b_2(n_f=4) = 2.08333 \ , \quad g_1^{(1)} = 8/3 \ , \quad g_2^{(1)} = -2 \ .$$

The optimum N, determined numerically, are shown in table 1.

Several details concerning the numerical calculations may be found in the appendix. Notice from eqs. (3.11), (3.12) that, for a given power ρ of the coupling constant, the maximum power of $\ln n$ in each of the three sums has coefficients

$$
\begin{aligned}
s^L_{\rho+1,\rho}[2I(t/2)] &= g_1^{(1)} \frac{b_2^{\rho-1}}{\rho(\rho+1)} 2^\rho \ , \\
s^L_{\rho+1,\rho}[I(t)] &= -g_1^{(1)} \frac{b_2^{\rho-1}}{\rho(\rho+1)} \ , \\
s^L_{\rho,\rho}[I_2(t)] &= g_2^{(1)} \frac{b_2^{\rho-1}}{\rho} \ .
\end{aligned}
\tag{4.1}
$$

TABLE 1

Optimum numbers of asymptotic terms as a function of n and α

α	n	$N[2I(t/2)]$	$N[I(t)]$	$N[I_2(t)]$
0.075	20	1	5	5
”	40	1	5	5
”	60	1	5	5
0.061	20	2	6	6
”	40	2	6	6
”	60	2	6	7
0.039	20	4	13	11
”	40	4	12	11
”	60	4	9	21

From table 1 we see that the leading power of $\ln n$ comes by far from the $N[I(t)]$ and the $N[I_2(t)]$ terms, and the corresponding coefficients, as shown in eq. (4.1) are both negative. This shows that the leading exponent, expressed perturbatively as an asymptotic approximation, becomes negative for sufficiently large values of the moment n. In momentum space this shows that, within perturbation theory, the corresponding cross section tends to a finite limit as $z \to 1$, since the hard part is the exponential of a quantity that diverges to minus infinity in that region of phase space. Of course, our asymptotic approximation is not valid at the edge of phase space. The preceding discussion, however, serves to point out that the corresponding resummed perturbative cross section is finite in that range, and

in no need of extra IR cut-offs. What is more important, we can reproduce this satisfactory feature of the hard part without even restricting ourselves to the perturbative regime. Indeed, we may numerically calculate the leading exponent either from eqs. (2.27)-(2.30), or from their analytical equivalent, eq. (2.36). The calculation is carried out in moment space, but conclusions can be reached in momentum space with the correspondence[2]

$$n \leftrightarrow \frac{1}{1-z} \; . \tag{4.2}$$

The calculation of $E(n,\alpha)_L$ from the series in eq. (2.36) is pretty sensitive to the accuracy with which intermediate quantities are calculated, because there are large cancelations involved. Alternately, as we explain in the appendix, it is convenient to evaluate the integrals in eqs. (2.28)-(2.30) numerically on a deformed version of the principal value contour, which is much more stable from a numerical point of view. This contour, denoted by $\bar{P}$, is shown in fig. 1. Some details of the integration may be found in the appendix. Using these integrals in eq. (2.27), we find the solid curves of fig. 2. On the other hand, we may use the asymptotic expression, eqs. (3.11), (3.12), together with values of N like those in table 1, to obtain a perturbative asymptotic series for the leading exponent. The corresponding values of $E(n,\alpha,N)_L$ as a function of n are shown by the dotted curves of fig. 2.

The first thing we note is that the exponent is bounded at fixed Q^2, and that it reaches its maximum at rather large values of n. Beyond the maximum, it decreases monotonically toward $-\infty$, due to the dominance in E of the integral $I_1(t)$, eq. (2.29), which behaves as $-t\ln(2)\ln(n)$ when $\ln(n) \gg t$. Comparison between the solid and dotted curves will establish the range of moments n where the perturbative expression is valid, as a function of the fixed coupling constant. Below its maximum, $E(n,\alpha)_L$ is nicely approximated by its asymptotic series, also shown in fig. 2. Beyond this, higher twist takes over and damps the exponent, eventually making it negative. The asymptotic series behaves in an analogous manner, but decreases only for much larger n. In either case, the limit of the exponent is minus infinity as $z \to 1$, and the integral over z in the cross section is actually finite for both the full principal value exponent and its asymptotic expansion constructed as above.

Introducing the notation

$$n_1 \equiv n\Big(E(n,\alpha)_L = \max\Big) \;, \quad n_2 \equiv n\Big(E(n,\alpha,N)_L = \max\Big)$$

and, with Δ_L given by eq. (3.10),

$$\delta_L(n,\alpha) \equiv \frac{|\Delta(n,\alpha,N)_L|}{E(n,\alpha)_L} \times 100\% \ ,$$

we may summarize the main features of this comparison in the following table:

TABLE 2

The exact exponent versus its asymptotic approximation

α	n_1	$E(n_1,\alpha)_L$	$E(n_1,\alpha,N)_L$	$\delta_L(n_1,\alpha)$	n_2	$E(n_2,\alpha,N)_L$
0.075	35	1.847	1.710	7.4%	680	3.833
0.061	100	3.005	4.266	41.9%	98×10^3	27.77
0.039	1880	6.635	10.954	65.1%	34×10^6	177.5

From table 2 we conclude that the asymptotic perturbative approximation is good nearly up to n_1, even though the error $\delta_L(n_1,\alpha)$ increases significantly with decreasing α. Beyond n_1, the two curves are numerically very different, showing that the higher-twist implicit in the solid curves of fig. 2, dominates. Notice also from fig. 2 that, for n fixed and below n_1, the error decreases (slowly) with decreasing α, something expected from the nature of the asymptotic series. For example, $\delta_L(35,0.061) = 2.6\%$ and $\delta_L(35,0.039) = 2.5\%$. Finally, we observe from table 2 that, for all three values of α chosen, $\alpha b_2 \ln n_1 \simeq 0.6$, while $\alpha b_2 \ln n_2 \simeq 1 - 1.4$.

5. Summary and Discussion

In this paper we have proposed a definition that removes the perturbative ambiguities of the QCD resummation formula, through a principal value prescription for integrals over the running coupling in the exponent E, defined in eq. (1.4). We have developed this approach in the context of the Drell-Yan cross section, but it should have a more general application.

This resummation procedure provides an unambiguous (although arbitrary[3]) definition for the resummed perturbative series for large threshold corrections in hadron-hadron

scattering, to which higher-twist nonperturbative corrections may, in principle, be added systematically. We have noted the relation of our principal value prescription to the method of Borel transformations.

The exponent $E(n, \alpha)$ is analytically calculable, in terms of one and two-loop effects that contain all the large perturbative corrections in the sense discussed in ref. 2. For the one-loop terms of the exponent, these results are given as an infinite series of Exponential Integrals. This series suggests, in turn, a definition of the resummed perturbative series, exploiting the asymptotic approximation of these functions. The number of terms in the asymptotic exponent is now precisely determinable numerically, as well.

Numerical results show that the exact leading exponent can enhance the cross section in a manner that is nicely approximated by its asymptotic perturbative expansion up to values of n such that $(\alpha_s(Q^2)/\pi) b_2 \ln n \simeq 0.6$. Beyond this range higher-twist effects turn the exact exponent around to negative values.

We are hopeful that this approach will be useful in organizing QCD perturbative corrections. In particular, the results for the exponent shown in fig. 2 suggest that resummed higher-order radiative corrections may be substantially smaller than the simple exponentiation of leading-order results. This may help to explain the relative success of one-loop approximations at moderate energies. In addition, we have seen that nonperturbative effects, whose presence is implied by ambiguities in resummed perturbation theory, appear at order Λ/Q. For the normalization of the Drell-Yan cross section then, we may be in a situation where perturbative corrections are moderate, with relatively large, but kinematically simple, nonperturbative corrections. Such a combination could teach us a lot about the interrelation of these two components of the full theory.

Of course, the analytical result for the leading exponent, eq. (2.36), contains only an (arbitrary) part of the physical higher twist in dilepton production, and we have not made a systematic study of higher twist in this paper. A definition of the perturbative series in QCD is, however, necessary before higher twist can be unambiguously defined[3] . Because eq. (2.36) is such a well-defined extension of the perturbative series for large corrections, it also contains *some* higher twist. This is evident numerically in the difference between the functions and their perturbative approximations at the edge of phase space in fig. 2. A complete analysis of higher twist in the normalization of the Drell-Yan cross section

is now possible, but must remain the subject of future work. One technical observation along these lines may be worth making here, however. In addition to the corrections at order $(\Lambda/Q)^p, p \geq 1$ that we have found above, we also expect corrections of the form $1/[(1-z)Q^2]$, associated, for instance, with the production of resonances in the final state. Such corrections may be equally, or even more important than those identified here, especially for very large Q.

In future work we hope to explore further the numerical and phenomenological consequences of this approach, including numerical results for the next-to-leading exponent, and for the cross section itself. This treatment of large perturbative corrections should also be applicable to other inclusive hadron-hadron cross sections, including heavy-quark production[12] and jet production, where gluon-gluon hard-scattering functions play an important role.

ACKNOWLEDGEMENTS

We would like to thank L. Alvero, G. Levin, A. Mueller, J. Smith and W. van Neerven for very helpful discussions. This work was supported in part by the National Science Foundation under grant PHY 9211367 and by the Texas National Research Laboratory.

APPENDIX A
Numerical Evaluation of the Exponent

The first issue we will address is the number of asymptotic terms in the formula giving $E(n, \alpha_s, N)_L$. As was mentioned in section 3, one definition could involve a comparison of each function $\mathcal{E}(mt)$ with its asymptotic expansion, eq. (3.2). Given that this function is tabulated and close to 1 at this range, this would seem a very straightforward approach. However, the value of N thus determined would depend on mt and one would have then to sum that complicated dependence over m. We took the simpler approach of using eq. (3.3) to define the three numbers $\{N[2I(t/2)],\ N[I(t)],\ N[I_2(t)]\}$ for each of the individual sums. These numbers of course now depend on the moment variable n, but not strongly. As we show in table 1, this dependence becomes more appreciable when the coupling constant decreases, but this has no numerical consequences for the leading asymptotic exponent in the perturbative regime, precisely because of the smallness of the corresponding coupling. The above numbers should be determined in a range of values of n that is well within the perturbative restriction $\alpha_s \ln n \ll 1$. The result is shown in table 1.

Another issue is the numerical evaluation of the exact leading exponent. If we use the simple and straightforward analytic expression, eq. (2.36), we observe that the series involved* , although convergent, are alternating. Using double precision, one can not reliably calculate the exponent beyond a value $n \simeq 40$, because the alternating partial sums exceed the 16-digit accuracy. Also, the special functions $\mathcal{E}(mt)$ should be calculated to a similar accuracy, and the corresponding numerical integrations need a lot of computer time. This method, for relatively low values of n, may be used as a numerical check for the alternative method we mentioned in section 4, namely the use of contour $\bar{P}$, fig. 1. The resulting expressions for the integrals are rather complicated, from an analytical point of view, but run smoothly on the computer for much larger values of n. On $\bar{P}$, the integral $I(t)$, eq. (2.29), can be written as a sum of three integrals, each one resulting from twice the real part of the integration along one side of the contour. The result is

$$I(t) = I_1(t, r) + I_2(t, r) + I_3(t, r) \tag{A.1}$$

* more accurately, the finite sums involved, if we use integer n (which we will do in this case),

with $r \equiv n-1$, where:

$$I_1(t,r) = t\int_0^1 \frac{dx}{x}\Big(A_1(r;x)B_1(t,r;x) + C_1(r;x)D_1(t,r;x)\Big) \tag{A.2}$$

$$I_2(t,r) = t\int_0^1 \frac{dx}{x^2+1/r^2}\Big(A_2(r;x)B_2(t,r;x) + C_2(r;x)D_2(t,r;x)\Big) \tag{A.3}$$

$$I_3(t,r) = -t\int_0^{1/r} \frac{dx}{1+x^2}\Big(A_3(r;x)B_3(t;x) + C_3(r;x)D_3(t;x)\Big) \tag{A.4}$$

and where the various integrands are given by:

$$A_1(r;x) = \Big(1+x^2/r^2\Big)^{r/2}\cos(r\arctan(x/r)) - 1 \tag{A.5}$$

$$B_1(t,r;x) = (1/2)\ln\Big(\Big[1+(1/t)\ln(x/r)\Big]^2 + \Big[\pi/2t\Big]^2\Big) \tag{A.6}$$

$$C_1(r;x) = \Big(1+x^2/r^2\Big)^{r/2}\sin(r\arctan(x/r)) \tag{A.7}$$

$$D_1(t,r;x) = \arctan\Big(\frac{\pi/2}{t+\ln(x/r)}\Big) \tag{A.8}$$

$$\begin{aligned} A_2(r;x) = x\Big(&\Big[(1-x)^2+1/r^2\Big]^{r/2}\cos\Big(r\arctan\Big(\frac{1/r}{1-x}\Big)\Big) - 1\Big) \\ &- (1/r)\Big[(1-x)^2+1/r^2\Big]^{r/2}\sin\Big(r\arctan\Big(\frac{1/r}{1-x}\Big)\Big) \end{aligned} \tag{A.9}$$

$$B_2(t,r;x) = (1/2)\ln\Big(\Big[1+(1/2t)\ln(x^2+1/r^2)\Big]^2 + \Big[\frac{\arctan(1/xr)}{t}\Big]^2\Big) \tag{A.10}$$

$$C_2(r;x) = (1/r)\Big(\Big[(1-x)^2 + 1/r^2\Big]^{r/2}\cos\Big(r\arctan\Big(\frac{1/r)}{1-x}\Big)\Big) - 1\Big)$$
$$+\, x\Big[(1-x)^2 + 1/r^2\Big]^{r/2}\sin\Big(r\arctan\Big(\frac{1/r}{1-x}\Big)\Big) \tag{A.11}$$

$$D_2(t,r;x) = \arctan\Big(\frac{\arctan(1/rx)}{t + (1/2)\ln(x^2 + 1/r^2)}\Big) \tag{A.12}$$

and

$$A_3(r;x) = x(x^r\cos(r\pi/2) - 1) + x^r\sin(r\pi/2) \tag{A.13}$$

$$B_3(t;x) = (1/2)\ln\Big(\Big[1 + (1/2t)\ln(1+x^2)\Big]^2 + \Big[\frac{\arctan x}{t}\Big]^2\Big) \tag{A.14}$$

$$C_3(r;x) = -(x^r\cos(r\pi/2) - 1 - x^{r+1}\sin(r\pi/2)) \tag{A.15}$$

$$D_3(t;x) = \arctan\Big(\frac{\arctan x}{t + (1/2)\ln(1+x^2)}\Big)\ . \tag{A.16}$$

The integral I_2 is handled in a similar fashion.

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FIGURE CAPTIONS

Figure 1: The contour $\bar{P}$ used for the numerical evaluation of the leading exponent. A mirror-symmetric contour in the lower half-plane is understood.

Figure 2: (a) The exact principal value leading exponent (solid curve) versus its asymptotic approximation (dotted curve) as a function of n, for $\alpha(Q = 5GeV) \simeq 0.075$. (b) Similarly for $\alpha(Q = 10GeV) \simeq 0.061$. (c) Similarly for $\alpha(Q = 90GeV) \simeq 0.039$.

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